A Parametric Solution to Common Tangents

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Abstract

We develop an efficient algorithm for the construction of common tangents between a set of Bezier curves. Common tangents are important in visibility, lighting, robot motion, and convex hulls. Common tangency is reduced to the intersection of parametric curves in a dual space, rather than the traditional intersection of implicit curves. We show how to represent the tangent space of a plane Bezier curve as a plane rational Bezier curve in the dual space, and compare this representation to the hodograph and the dual Bezier curve. The detection of common tangents that map to infinity is resolved by the use of two cooperating curves in dual space, clipped to avoid redundancy. We establish the equivalence of our solution in dual space to a solution in Plücker space, where all the same issues are encountered in a higher-dimensional context.

1 Introduction

The common tangents of two curves are at the heart of many problems involving visibility. Consider a collection of closed curves in the plane, interpreted as obstacles for a robot. The common tangents of these curves define the limits of visibility, and consequently the shortest paths that can be travelled by the robot (Figure 1). If we instead interpret the collection of closed curves as an area light source and occluders, the common tangents define the boundaries of the shadows (umbra and penumbra) that are cast by the light (Figure 2). Common tangents are also needed for basic geometric computations with curves, such as the convex hull. [11] mentions many more applications, including font design.

The common tangency problem is truly an intersection problem, since a common tangent is simultaneously a tangent of two curves. The standard solution reduces common tangency to the intersection of implicit curves. We explore an alternate reduction, to the intersection of parametric Bezier curves. The robustness, efficiency, and simplicity of Bezier intersection make this an attractive solution.

Let \( C(s) = (c_x(s), c_y(s)) \) and \( D(t) = (d_x(t), d_y(t)) \)
Figure 3. A tangent space is a family of lines

be plane curves of degree \( n \). The standard solution [1, 11] argues as follows. Since a line between \( C(s) \) and \( D(t) \) is a common tangent if and only if it is orthogonal to the normals at both \( C(s) \) and \( D(t) \), a common tangent is a parameter pair \( (s, t) \) satisfying the following system of equations:

\[
\begin{align*}
(c_x(s) - d_x(t), c_y(s) - d_y(t)) \cdot (-c_y'(s), c_x'(s)) &= 0 \\
(c_x(s) - d_x(t), c_y(s) - d_y(t)) \cdot (d_y(t), d_x(t)) &= 0
\end{align*}
\]

This is the intersection of two implicit curves of degree \( 2n - 1 \) in parameter space:

\[
\begin{align*}
f(s, t) &= 0 \\
g(s, t) &= 0
\end{align*}
\]

In our solution, we interpret the tangent space of a curve as a family of lines (Figure 3), and a common tangent as an intersection of the line families of two curves. To simplify the intersection of line families, we map to a dual space. By encoding a tangent line as a point in a dual space, a family of tangent lines becomes a parametric curve in the dual space (Figure 4). The computation of common tangents then reduces to the intersection of curves in dual space (Figure 5). The representation of tangent space as a rational Bezier curve in dual space (Section 2) is a development of independent interest. It can be a useful tool for the analysis of tangent spaces.

Since our solution involves tangent spaces, dual spaces, and point representations of a line, there is much related work to explore. We choose to present the new method in its entirety first, and make comparisons afterwards, so that our method is entirely clear and an informed comparison can be made.

The paper is organized as follows. Section 2 introduces the dualization of a line and the representation of a plane curve’s tangent space by a plane curve in the dual space, the tangential curve. This section also shows how to express the tangential curve as a rational Bezier curve. Section 3 deals with the issue of points at infinity: it is inevitable that some tangents map to infinity in the dual space, and we ensure that we find all common tangents by replacing a single tangential curve by two cooperating, clipped tangential curves. This section also includes the final version of our algorithm. Experimental results are given in Section 4. In the remaining sections, we relate our work to classical work on tangent spaces, common tangents, dual spaces, and point representations of a line. Section 5 analyzes the other solutions to common tangents and the advantages of our parametric solution in more detail. Section 6 explains the inadequacy of the hodograph for common tangents. Section 7 discusses other work on duality, including the dual Bezier curve. Finally, Section 8 establishes the basic equivalence of our solution with a Plücker coordinate solution. Section 9 wraps up with ideas for future work. An appendix fills in some necessary technical detail on tangential curves.

2 A representation of tangent space

To map a tangent space to a curve, we must first be able to map a line to a point.

**Definition 2.1** The line \( ax + by + c = 0 \) in 2-space is dual to the point \((a, b, c) \in \mathbb{P}^2\) in projective 2-space.

The dual point must live in projective space, since the dual \((a, b, c)\) must be equivalent to the dual \((ka, kb, kc)\), as the line \( ax + by + c = 0 \) is equivalent to the line \( kax + kby + kkc = 0 \), \( k \neq 0 \).

Since the tangent space of a plane curve is a continuous one-dimensional family of lines, a tangent space can now be represented by a curve in dual space.
Definition 2.2 Let $C(t)$ be a plane curve. The tangential curve of $C(t)$ is the curve $C^*(t) \subset P^2$, where $C^*(t)$ is the dual of the tangent at $C(t)$.

Figure 4 gives a simple example of a tangential curve, and Figure 5 illustrates how the computation of common tangents in primal space becomes curve intersection in dual space.

We now express the tangential curve as a Bezier curve, first as a Bezier curve in projective space, and then as a rational Bezier curve. We begin by developing the parametric form of the tangential curve for an arbitrary parametric input curve, and then the Bezier form for a Bezier input curve. This result also applies to B-spline curves and polynomial parametric curves, which can be easily expressed as Bezier curves. The Bezier form of the tangential curve for a rational Bezier input curve is a more elaborate application of the same methods.

Theorem 2.3 Let $C(t)$ be a plane Bezier curve of degree $n$ with control points $\{b_i\}_{i=0}^n$ over the parameter interval $[t_1, t_2]$. The tangential curve $C^*(t)$ is a Bezier curve of degree $2n - 1$ in projective 2-space with control points

$$
eq \alpha \sum_{j=m}^{n-1} \sum_{i=0}^{n-1} \binom{n-1-i}{j-i} \binom{n-j-1}{i} (b_{j-i} - b_{j-i+1}) \delta_{i,j}$$

$k = 0, \ldots, 2n - 1$, $b_k^* \in P^2$, over the parameter interval $[t_1, t_2]$, where $\alpha = \frac{1}{\Delta}$, $\Delta = t_2 - t_1$, and $\beta_{i,j} = (b_{i+1,2} - b_{i+1,1})b_{j,1} - (b_{i+1,1} - b_{i+1})b_{j,2}$.

Proof: Since the proof is rather technical, the majority of it is postponed to the appendix. We only develop the basic parametric representation here. Let $C(t) = (c_1(t), c_2(t))$. The tangent and normal vectors of $C(t)$ are $(c'_1(t), c'_2(t))$ and $(-c'_2(t), c'_1(t))$, respectively. Since the implicit equation of the line through $P$ with normal $N$ is $(X - P) \cdot N = 0$, the implicit equation of the tangent of $C(t)$ is:

$$(x - c_1(t), y - c_2(t)) \cdot (-c'_2(t), c'_1(t)) = 0$$

or

$$-c'_2(t)x + c'_1(t)y + [c'_2(t)c_1(t) - c'_1(t)c_2(t)] = 0$$

and the tangential curve of $C(t)$ is

$$(-c'_2(t), c'_1(t), c'_2(t)c_1(t) - c'_1(t)c_2(t))$$

(2)

This is a polynomial curve in projective 2-space. ■

A Bezier curve in projective 2-space is equivalent to a rational Bezier curve in 2-space. We rephrase Theorem 2.3 accordingly.

Corollary 2.4 Let $C(t)$ be a plane Bezier curve of degree $n$ with control points $\{b_i\}_{i=0}^n$ over $[t_1, t_2]$. The tangential curve $C^*(t)$ is a rational Bezier curve of degree $2n - 1$ over $[t_1, t_2]$ with weights $\{w_k\}_{k=0}^{2n-1}$ where:

$$w_k = \alpha \sum_{j=m}^{n-1} \sum_{i=0}^{n-1} \binom{n-1-i}{j-i} \binom{n-j}{i} \beta_{i,j}$$

and control points $\{c_k\}_{k=0}^{2n-1}$ where:

$$c_k = \frac{\alpha}{w_k} \sum_{j=m}^{n-1} \binom{n-1-i}{j-i} \binom{n-j}{i} (b_{j,i} - b_{j,i+1})$$

where $\alpha$ and $\beta_{i,j}$ are as in Theorem 2.3.

3 Recovering intersections at infinity

Lines $ax + by = 0$ through the origin map to points at infinity $(a, b, 0)$ in dual space. As a result, if two curves have a common tangent through the origin, the associated intersection point in dual space will be at infinity, and effectively will not be found. In practice, all common tangents ‘nearly’ through the origin will also be lost. How can we correct this?

Our solution is to replace a tangential curve by two cooperating, clipped tangential curves, as follows. In mapping the line $ax + by + c = 0$ to the point $(a, b, c)$ in projective 2-space, any of the coordinates of $(a, b, c)$ may be considered the projective coordinate. We have chosen $c$ as the projective coordinate, which results in mapping lines $ax + by = 0$ to infinity. If we instead choose $a$ as the projective coordinate, horizontal lines $by + c = 0$ are mapped to infinity, or if we choose $b$, vertical lines $ax + c = 0$ are mapped to infinity.

Definition 3.1 Let $\alpha$ be a plane curve. The (tangential) a-curve (resp., b-curve or c-curve) of $\alpha$ is the image of the tangent space of $\alpha$ in dual space when $a$ (resp., $b$ or $c$) is chosen as the projective coordinate in dual space.
Thus, the tangential curve of Corollary 2.4 is a \(c\)-curve, and horizontal tangents are mapped to infinity on the \(a\)-curve, while vertical tangents are mapped to infinity on the \(b\)-curve.

Since a line cannot be simultaneously horizontal and vertical, any tangent will be a finite point either on the \(a\)-curve or the \(b\)-curve. Thus, no common tangents associated with intersections at infinity will be missed if we compute both the intersections of the tangential \(a\)-curves and the intersections of the tangential \(b\)-curves.\(^2\) This replaces the earlier intersection of \(c\)-curves only. (See Figures 6-9.)

The \(a\)-curves and \(b\)-curves still duplicate effort. If a common tangent is found as an intersection of the \(a\)-curves, we do not want it also to be found as an intersection of the \(b\)-curves, for the sake of efficiency. The solution is to make the tangential \(a\)-curve represent more vertical tangents \((ax + by + c = 0\) with \(|a| \geq |b|\)) while the tangential \(b\)-curve represents more horizontal tangents \((ax + by + c = 0\) with \(|a| \leq |b|\)). A common tangent that is more vertical will be found as an intersection of the \(a\)-curves, while a common tangent that is more horizontal will be found as an intersection of the \(b\)-curves, making for a perfect division of duty. We find the diagonal (45- and 135-degree) tangents, which separate the more-horizontal from the more-vertical tangents, and then clip the curves at these tangents. Notice that this clipping is equivalent to a clipping of segments about the points at infinity.

### 3.1 The final algorithm

The use of 2 curves and clipping leads to the following algorithm for computing the common tangents of \(C\) and \(D\), as illustrated by Figure 6:

1. **Compute the tangential \(a\)-curves** \(C^*_a\) and \(D^*_a\) and the tangential \(b\)-curves \(C^*_b\) and \(D^*_b\).

2. **Clip at diagonal tangents**, as follows. Subdivide \(C^*_a\) and \(C^*_b\) (resp., \(D^*_a\) and \(D^*_b\)) at the parameter values of the diagonal tangents of \(C\) (resp., \(D\)), and classify the resulting segments as more-vertical or more-horizontal, by testing an interior tangent. Clip out the more-horizontal segments from \(C^*_a\) and \(D^*_a\) and the more-vertical segments from \(C^*_b\) and \(D^*_b\).

3. **Intersect the clipped** \(C^*_a\) and \(D^*_a\) in \(a\)-space (the dual space of the \(a\)-curves), and the clipped \(C^*_b\) and \(D^*_b\) in \(b\)-space.

4. **Map both sets of intersections in dual space to common tangents in primal space.**

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\(^2\)Since a line can simultaneously be horizontal and pass through the origin, or vertical and pass through the origin, combining the intersections of tangential \(a\)-curves and \(c\)-curves, or \(b\)-curves and \(c\)-curves, is not useful. We would still miss horizontal or vertical common tangents through the origin.

The common tangents of a single curve can also be computed using this algorithm (Figure 9). The only difference is that \(C\) and \(D\) are the same curve, and the intersection of step (3) is self-intersection.

In many applications (e.g., motion planning and lighting), visible common tangents are the only common tangents of interest, and an extra filtering step should be added to the algorithm (Figure 2). A common tangent is visible if its two points of tangency are visible (i.e., the line segment connecting them has no interior intersections with any obstacles). Since a common tangent is by definition close to the obstacle as it approaches it, and the intersection algorithm is approximate, care must be taken in evaluating visibility.

The computation in each step of the algorithm is efficient. Consider each step:

1. The appendix gives formulae for computing tangential \(a\)-curves and \(b\)-curves.

2. The diagonal tangents of the curve \(a\) are the tangents \(ax + by + c = 0\) with \(|a| = |b|\). The parameter values of these diagonal tangents can be found by intersecting the tangent hodograph of \(a\) with the diagonal lines \(x = \pm y\).

3. This is traditional Bezier curve intersection [13]. For the common tangents of a single curve, it is Bezier self-intersection [8].

4. Rather than using duality, the following solution is preferable to map the intersections back to common tangents. We record each intersection as a parameter pair \((s, t)\), the parameter values of the intersection point with respect to each curve. The two endpoints \(C(s)\) and \(D(t)\) of the common tangent are then immediately available. These endpoints are necessary for testing if a common tangent is visible.

The clipping by diagonal tangents may also be interpreted as a clipping of the tangential curves by the lines \(x = \pm 1\) (see Figures 7-8). In particular, the diagonal lines in primal space map to the vertical lines \(x = \pm 1\) in both dual \(a\)-space and dual \(b\)-space: \(ax \pm ay + c = 0\) maps to \((\frac{x}{a}, \frac{y}{a}) = (\pm 1, y), \ y \in \Re\). Segments inside \(x \in [-1, 1]\) should be kept.

### 3.2 The inevitability of intersections at infinity

The development of this section may prompt the reader to ask if it is possible to reduce common tangency to the intersection of plane curves without introducing the problem of certain lines mapping to infinity. The answer is no. The mapping of some lines to infinity is a necessary consequence of stuffing the 3 dimensions of the family of lines.
ax + by + c = 0 into the 2 dimensions of a projective 2-space. We must stuff the 3 dimensions of tangent space into a 2-dimensional dual space, since common tangency will not reduce to intersection if we represent the tangent space as a space curve in 3 dimensions (space curves do not intersect in general). Moreover, this 2-dimensional dual space is inherently projective, since \( ax + by + c = 0 \) is equivalent to \( kax + kby + kc = 0 \). Thus, one of the three coordinates of a tangent line must be considered the projective coordinate, and when this coordinate goes to 0, we have a point at infinity. For example, we will later develop a Plücker version of the tangential curve that is also 3-dimensional and has the same problems with points at infinity (Section 8).

3.3 Clipping and negative weights

Looking at (3), it is clear that some of the weights of a tangential curve can be negative. Negative weights are undesirable, since curve segments with negative weights lack the convex hull property that is so useful for divide and conquer techniques of intersection using subdivision. Goldman and DeRose [4] have shown how to intersect curves without the convex hull property, using an expanded convex hull, but we have no need for this special intersection algorithm.

The use of clipped tangential curves in the final algorithm has an added benefit: it removes the problem of negative weights. The clipping of more-horizontal tangents from a tangential \( a \)-curve, and more-vertical tangents from a \( b \)-curve, is equivalent to the clipping of a neighbourhood about all points at infinity (zero weights). This is equivalent to the clipping of all sign changes of the weight. The weights of the remaining segments are purely positive or purely negative. The negative weights on a segment with purely negative weights can be easily removed by multiplying all the weights by \(-1\), since rational Bezier curves are invariant under transformations \( w_i = kw_i \) of the weights. Thus, all weights remaining on the clipped tangential curves are positive.

The issue of negative weights is another indication that the use of two clipped curves is a natural solution to the robust reduction of common tangency to the intersection of rational Bezier curves.

4 Results

We would like to illustrate the algorithm on some examples (Figures 6-9). In all of these examples, on the left we show a pair of curves and their common tangents, while on the right we show the dual world of the clipped tangential \( a \)-curves and \( b \)-curves and their intersections. All common tangents are shown, visible and invisible. Figure 2 shows an example of visible common tangents only. Figure 6 is a simple example to introduce the method. Figures 7-8 are more challenging examples. Notice how the common tangents in Figure 8 are still found, and still computed robustly, even in high curvature regions of the curves, where the problem is fragile and ill-conditioned. Figure 9 shows the computation of the common tangents between a single curve. The curves and their tangential curves are the same as in Figure 7: we have simply zoomed in on the tangential curves to better show detail near the intersections.

5 Other work on common tangents

In the introduction, we saw that the classical solution to the common tangents of two parametric curves \( C(s) \) and \( D(t) \) of degree \( n \) reduces to the intersection of two implicit curves \( f(s, t) = 0 \) and \( g(s, t) = 0 \) of degree \( 2n - 1 \). The symbolic solution of this system using resultants is expensive, since it isolates the coordinates of the solutions independently, and reduces to the filtering of \( O(16n^5) \) parameter pairs (all combinations of \( O(4n^2) \) \( s \)- and \( t \)-coordinates) through the evaluation of the curves \( C(s) \) and \( D(t) \) and their hodographs \( C'(s) \) and \( D'(t) \) at each parameter pair. The resultant solution also has difficulty with the Bezier structure of the curves. A numerical solution using techniques of Nishita et. al. [9] or Sherbrooke and Patrikalakis [15] is more attractive, where the implicit curves are expressed in the Bernstein basis and intersected using Bezier subdivision techniques.

The contrast between the classical solution and our method is between an intersection of degree \( 2n - 1 \) implicit curves in \((s, t)\) parameter space and the intersection of degree \( 2n - 1 \) parametric Bezier curves in dual space. The parametric solution in dual space is appealing, redrafting the solution in terms of the more familiar intersection of Bezier curves. The intersections in dual space are significantly accelerated by their restriction to the range \( x \in [-1, 1] \). Moreover, the parametric solution in dual space generalizes naturally to a solution for common tangent planes of surfaces in 3-space, reducing to the intersection of two Bezier surfaces in 3-space, a simpler solution than the generalization of the classical solution to the intersection of several implicit surfaces in 4-space.

Another solution to common tangents has been proposed by Parida and Mudur [11], using a geometric divide-and-conquer approach. Their solution works as follows. The curves are first decomposed into ‘C-shaped curves’, convex segments monotone in tangent direction. Each pair of C-curves, one from each curve, is compared to determine if the pair can define a common tangent, as follows. The pair is first reparameterized and clipped so that the C-curves share the same parameter interval and have the same tangent directions at the endpoints. If certain rejection criteria involving the tangent range are satisfied, the pair is rejected. If the two C-curves are pseudo-linear, a test for a common tangent
Figure 6. A simple example

Figure 7. A more complicated example
Figure 8. Another complicated example

Figure 9. Common tangents of a single curve
is directly applied. Otherwise, both C-curves are subdivided at their shoulder points (the point whose tangent is parallel to the chord of the endpoints) and the process is repeated recursively. The subdivision involved in this solution is more complicated than the simple midpoint subdivision of Bezier intersection, requiring the calculation of a shoulder point at each subdivision and reparameterization. Moreover, an initial decomposition into C-curves is expensive, and there is more subdivision.

We observe that Sederberg and Nishita [14] solve a special case of the common tangent problem, showing how to compute the points of tangency of two plane Bezier curves (i.e., an intersection point of the curves where they share the same tangent).

6 Why not the hodograph?

Our solution works with the tangent space of a curve. In this section, we explore why we don’t use the standard representation of the tangent space of a curve, the hodograph. Unfortunately, the intersection of hodographs does not yield common tangents. The problem is that the hodograph represents a tangent vector, not a tangent line. A point $H(t)$ of the hodograph of a curve $C(t)$ represents the tangent vector at $C(t)$, so the tangent line at $C(t)$ is $C(t) + sH(t)$. There are two problems with this representation: (1) the tangent line depends on both the hodograph and the original curve, and (2) many points in the hodograph’s space represent the same tangent line: in particular, all points along the same line through the origin represent the same direction and thus the same tangent line. (2) means that a common tangent of two curves does not imply an intersection of their hodographs unless the tangent vectors at either end of the common tangent are exactly the same length. (1) means that, conversely, an intersection of the hodographs does not imply a common tangent, but just two parallel tangents. Thus, hodographs are not useful for finding common tangents.

7 Other work using duality

Our solution involves dual space, a popular tool for geometric analysis. In this section, we compare it to other work using duality. The abstract idea of dualizing tangent spaces into curves is introduced in [5, p. 54]. Dualities between a point and a hyperplane are often used in computational geometry [2, 10], usually for the translation of problems on configurations of points to problems on finite arrangements of hyperplanes.

A representation for plane curves using duality has been developed by Hoschek, called dual Bezier curves [6], where a plane curve is represented as the envelope of a set of lines rather than as a set of points. A line family is represented in terms of control lines $L_n$, and the curve is understood to be the envelope of this line family. A point of the curve can still be evaluated using a de Casteljau algorithm to refine the control lines. Pottmann gives a formula for translating between the Bezier curve (control points) and dual Bezier curve (control lines) representations [12]. While the dual curve is a curve in primal space expressed in a traditional control structure (control lines rather than control points) for the representation of plane curves, the tangential curve is a curve in dual space expressed in a traditional control structure for the analysis of tangent spaces. In particular, the dual Bezier curve is not appropriate for our intersection problem in dual space.

8 Equivalence of a Plücker solution

Our solution involves the mapping of lines to points. We have used the duality of Definition 2.1. Why don’t we use the classical Plücker representation of a line by a point? A solution to common tangency can indeed be developed using a mapping of tangent spaces to curves in Plücker space and intersection of these curves. It turns out that this Plücker solution is almost entirely equivalent. We use the dual space solution since its development is more natural for lines in 2-space, and leads to a slightly simpler implementation.

Our dual representation of a line only works in 2-space, since the line is a hyperplane (representable by a single implicit equation) in 2-space but not in 3-space. However, in 2-space, the dual representation is a bit simpler, more intuitive, and direct than Plücker coordinates. As a representation of any line in 3-space, Plücker coordinates are more general and Plücker space is a higher-dimensional version of our dual space: a 5-dimensional, rather than 2-dimensional, projective space. In all other ways, the solution of the common tangency problem in Plücker space is identical to our solution in dual space. In particular, a tangent space is again mapped to a curve in Plücker space, and the tangential curve in Plücker space is equivalent to the tangential curve in dual space. This can be seen as follows.

Definition 8.1 The Plücker tangential curve of $C(t)$ is the curve $C^{**}(t) \subset P^5$, where $C^{**}(t)$ are the Plücker coordinates of the tangent at $C(t)$.

The Plücker coordinates of the line $P + uV$ are $(V, P \times V)$. Since the tangent line at $C(t)$ is $C(t) + uC'(t)$, the Plücker tangential curve of $C(t)$ is the curve $(C'(t), C(t) \times C'(t))$. If $C(t) = (c_1(t), c_2(t), 0)$ is a plane curve, this Plücker tangential curve becomes

$$(c'_1, c'_2, 0, 0, c_1c'_2 - c'_1c_2)$$

(4)
in projective 5-space. Compare this to (2). This establishes the essential equivalence of the Plücker solution.

9 Conclusions

This paper explores the implementation of the following idea: common tangency in primal space as parametric curve intersection in dual space. The solution that we have developed is efficient, robust, and simple to implement. The ability to efficiently compute common tangents allows us to revisit problems in visibility, shortest path motion, and lighting in a smooth environment composed of smooth obstacles or smooth area lights. This work also establishes the foundations for working with tangent spaces in a dual space, which is a powerful paradigm for the analysis of tangent spaces of curves and surfaces. A natural extension to the calculation of common tangent planes of smooth surfaces is being explored.

10 Acknowledgements

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References


11 Appendix

Proof of Theorem 2.3: Recall that we have established that the tangential curve of \( C(t) \) is

\[
(-c_2'(t), \ c_1'(t), \ c_2'(t)c_1(t) - c_1'(t)c_2(t))
\]

(5)

We want to express the tangential curve as a Bezier curve. Consider the third coordinate of the tangential curve, which can be expressed as:

\[
\frac{1}{\Delta} \sum_{i=0}^{n-1} n \Delta b_{i+1} B^n_{i}^{-1}(t) \sum_{j=0}^{n} b_{j} B^n_{j}(t)
\]

\[
-\frac{1}{\Delta} \sum_{i=0}^{n-1} n \Delta b_{i+2} B^n_{i+1}^{-1}(t) \sum_{j=0}^{n} b_{j} B^n_{j}(t)
\]

where \( \Delta = t_2 - t_1 \) is the length of the parameter interval and \( \Delta b_{i+2} = b_{i+1,2} - b_{i,2} \). We can use the product formula for Bernstein polynomials [3] to simplify this to

\[
\frac{1}{\Delta} \sum_{i=0}^{n-1} \sum_{j=0}^{n} \frac{n-1}{r} \frac{n}{j} \frac{n-1}{r+j} \Delta b_{i+1,1} B^n_{r+j}(t) [\Delta b_{i+2} b_{j,1} - \Delta b_{i+1,1} b_{j,2}].
\]
Letting \( k = i + j \), this becomes
\[
\sum_{k=0}^{2n-1} B_k^{2n-1}(t) \sum_{0 \leq i \leq n-1, \ 0 \leq j \leq n, \ i+j = k} \Delta \binom{n-1}{i} \binom{n}{j} \beta_{i,j}
\]

The third coordinate is now expressed as a 1-dimensional Bezier curve of degree \( 2n - 1 \) with control points \( \{ b_{k,3} \}^{2n-1}_{k=0} \) where:
\[
b_{k,3} = \left( \frac{2n}{k} \right) \sum_{0 \leq i \leq n-1, \ 0 \leq j \leq n, \ i+j = k} \Delta \binom{n-1}{i} \binom{n}{j} \beta_{i,j}. \quad (6)
\]

Now consider the first two coordinates of the tangential curve (5),
\[
-c_2'(t) = \frac{1}{\Delta} \sum_{k=0}^{n-1} -n \Delta b_{k,2} B_k^{2n-1}(t) \quad (7)
\]

and
\[
c_1'(t) = \frac{1}{\Delta} \sum_{k=0}^{n-1} n \Delta b_{k,1} B_k^{2n-1}(t) \quad (8)
\]

These are 1-dimensional Bezier curves of degree \( n - 1 \), which we need to degree-elevate to degree \( 2n - 1 \) for compatibility with the third coordinate. A Bezier curve of degree \( n \) with control points \( \{ b_k \}^{n}_{k=0} \) is degree elevated [3] to a Bezier curve of degree \( n+r \) with control points \( \{ d_k^{[r]} \}^{n+r}_{k=0} \) where:
\[
d_k^{[r]} = \min(n,k) \sum_{j=\max(0,k-r)}^{\min(n,k)} d_j \binom{n}{j} \binom{r}{k-j} \binom{k}{k-j-r} \quad (9)
\]

Consequently, the 1-dimensional Bezier curve (7) of degree \( n - 1 \) with control points \( \Delta b_{k,2} \) is degree elevated to a 1-dimensional Bezier curve of degree \( 2n - 1 \) with control points \( \{ b_{k,2} \}^{2n-1}_{k=0} \) where:
\[
b_{k,2} = \min(n,k-1) \Delta b_{j,2} \left( \binom{n-1}{j} \binom{n}{j} \right) \frac{1}{\Delta} \binom{2n-1}{k}. \quad (10)
\]

Similarly, the Bezier curve (8) of degree \( n - 1 \) with control points \( \Delta b_{k,1} \) can be degree elevated to a Bezier curve of degree \( 2n - 1 \) with control points \( \{ b_{k,1} \}^{2n-1}_{k=0} \) where:
\[
b_{k,1} = \min(n,k-1) \Delta b_{j,1} \left( \binom{n-1}{j} \binom{n}{j} \right) \frac{1}{\Delta} \binom{2n-1}{k}. \quad (11)
\]

Combining (6), (9) and (10), we conclude that the tangential curve \( C^*(t) \) is a Bezier curve of degree 2\( n - 1 \) in projective space with control points \( \{ b_{k,3}^{2n-1} \}^{2n-1}_{k=0} \) as in (1).

We next give formulae for the tangential \( a \)-curves and \( b \)-curves. These are computed just like our original tangential curves, simply interpreting the projective coordinate anew.

Theorem 11.1 Let \( C(t) \) be a plane Bezier curve of degree \( n \) with control points \( \{ b_k \}^{n}_{k=0} \) over \([t_1, t_2]\). The tangential \( a \)-curve \( C^a(t) \) is a rational Bezier curve of degree \( 2n - 1 \) over \([t_1, t_2]\) with weights \( \{ w_k \}^{2n-1}_{k=0} \) where:
\[
w_k = \alpha \sum_{j=\max(0,k-n)}^{\min(n-1,k)} \beta_{i,j} \binom{n-1}{j} \binom{n}{j} \binom{2n-1}{k} \binom{b_{j,2} - b_{j+1,2}}{b_{j,1} - b_{j+1,1}}
\]

and control points \( \{ c_k \}^{2n-1}_{k=0} \) where:
\[
c_k = \alpha \sum_{j=\max(0,k-n)}^{\min(n-1,k)} \beta_{i,j} \binom{n-1}{j} \binom{n}{j} \binom{2n-1}{k} \binom{b_{j,2} - b_{j+1,2}}{b_{j,1} - b_{j+1,1}}
\]

where \( \alpha \) and \( \beta_{i,j} \) are as in Theorem 2.3.

Theorem 11.2 Let \( C(t) \) be a plane Bezier curve of degree \( n \) with control points \( \{ b_k \}^{n}_{k=0} \) over \([t_1, t_2]\). The tangential \( b \)-curve \( C^b(t) \) is a rational Bezier curve of degree \( 2n - 1 \) over \([t_1, t_2]\) with weights \( \{ w_k \}^{2n-1}_{k=0} \) where:
\[
w_k = \alpha \sum_{j=\max(0,k-n)}^{\min(n-1,k)} \beta_{i,j} \binom{n-1}{j} \binom{n}{j} \binom{2n-1}{k} \binom{b_{j,1} - b_{j+1,1}}{b_{j,2} - b_{j+1,2}}
\]

and control points \( \{ c_k \}^{2n-1}_{k=0} \) where:
\[
c_k = \alpha \sum_{j=\max(0,k-n)}^{\min(n-1,k)} \beta_{i,j} \binom{n-1}{j} \binom{n}{j} \binom{2n-1}{k} \binom{b_{j,1} - b_{j+1,1}}{b_{j,2} - b_{j+1,2}}
\]

where \( \alpha \) and \( \beta_{i,j} \) are as in Theorem 2.3.