Lecture 1.5: Proof Techniques

CS 250, Discrete Structures, Fall 2015

Nitesh Saxena

Adopted from previous lectures by Cinda Heeren

Course Admin

- Slides from previous lectures all posted
- HW1 posted
  - Due at 11am on Sep 17
  - Please follow all instructions
  - Questions?
Outline

- Proof Techniques

Main techniques

- Direct proofs
- Proofs using contraposition
- Proofs using contradiction
Direct Proofs: first example

Theorem: If $n$ is an odd natural number, then $n^2$ is also an odd natural number.

Proof:

If $n$ is odd, then $n = 2k + 1$ for some int $k$.

This means that:

$$n^2 = (2k+1)(2k+1)$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2j + 1$$ for some int $j$

$\Rightarrow$ $n^2$ is odd

Direct Proofs

An example:

Prove that if $n = 3 \mod 4$, then $n^2 = 1 \mod 4$.

HUH?

$7 = 3 \mod 4$

$7 = 111 \mod 4$

$37 = 1 \mod 4$

$37 = 61 \mod 4$

$94 = 2 \mod 4$

$94 = 6 \mod 4$

$16 = 0 \mod 4$

$16 = 1024 \mod 4$
Direct Proofs

Coming back to our Theorem:

If $n = 3 \mod 4$, then $n^2 = 1 \mod 4$.

Proof:

If $n = 3 \mod 4$, then $n = 4k + 3$ for some int $k$.

This means that:

$n^2 = (4k + 3)(4k + 3)$

$= 16k^2 + 24k + 9$

$= 16k^2 + 24k + 8 + 1$

$= 4(4k^2 + 6k + 2) + 1$

$= 4j + 1$ for some int $j$

$= 1 \mod 4$.

Proofs by Contraposition

Recall: Contrapositive: $p \rightarrow q$ and $\neg q \rightarrow \neg p$

Ex. “If it is noon, then I am hungry.”

“If I am not hungry, then it is not noon.”

We also know that: $p \rightarrow q = \neg q \rightarrow \neg p$

Therefore, if establishing a direct proof ($p \rightarrow q$) is difficult for some reason, we can instead prove its contraposition ($\neg q \rightarrow \neg p$), which may be easier.
Proofs by Contraposition: example

Theorem: If $3n + 2$ is an odd natural number, then $n$ is also an odd natural number.

Proof:

If $n$ is not odd, then $n = 2k$ for some int $k$.

This means that:

$3n+2 = 3(2k) + 2$
$= 2(3k) + 2$
$= 2(3k + 1)$
$= 2j$ for some int $j$

$\Rightarrow 3n+2$ is not odd

Proofs by Contraposition: another example

Theorem: If $N = ab$ where $a$ and $b$ are natural numbers, then $a \leq \sqrt{N}$ or $b \leq \sqrt{N}$.

Proof:

If $a > \sqrt{N}$ AND $b > \sqrt{N}$, then by multiplying the two inequalities, we get

$ab > N$

This negates the proposition $N=ab$
Proofs by Contradiction

Recall: Contradiction is a proposition that is always False.

To prove that a proposition $p$ is True, we try to find a contradiction $q$ such that $\neg(p \rightarrow q)$ is True.

If $\neg(p \rightarrow q)$ is True and $q$ is False, it must be the case that $p$ is True.

We suppose that $p$ is False and use this to find a contradiction of the form $r \land \neg r$.

Proof:

A: $n$ is prime
B: $n$ is odd

We need to show that $A \rightarrow B$ is true.

We need to find a contradiction $q$ such that: $\neg (A \rightarrow B) \rightarrow q$

We know: $\neg (A \rightarrow B) \equiv \neg (\neg A \lor B) \equiv A \land \neg B$

This means that we suppose $(n$ is prime) AND $(n$ is even) is True.

But, if $n$ is even, it means $n$ has 2 as its factor, and this means that $n$ is not prime.

This is a contradiction because $(n$ is prime) AND $(n$ is not prime) is True.
Proofs by Contradiction: example

**Theorem:** If $3n + 2$ is an odd natural number, then $n$ is also an odd natural number.

**Proof:**

A: $3n + 2$ is odd

B: $n$ is odd

We need to show that $A \rightarrow B$ is true

We need to find a contradiction $q$ such that: $\neg (A \rightarrow B) \rightarrow q$

We know: $\neg (A \rightarrow B) \equiv \neg (\neg A \lor B) \equiv A \land \neg B$

This means that we suppose that $(3n + 2$ is odd) AND $(n$ is even) is True.

But, if $n$ is even, it means $n = 2k$ for some int $k$, and this means that $3n + 2 = 6K+2 = 2(3K+1) \rightarrow$ even.

This is a contradiction: $(3n + 2$ is odd) AND $(3n +2$ is even)

---

Disproving something: counterexamples

If we are asked to show that a proposition is False, then we just need to provide one counter-example for which the proposition is False

In other words, to show that $\forall x P(x)$ is False, we can just show $\neg \forall x P(x) = \exists x \neg P(x)$ to be True

Example: “Every positive integer is the sum of the squares of two integers” is False.

Proof: counter-examples: 3, 6,...
Today’s Reading

- Rosen 1.7
- Please start solving the exercises at the end of each chapter section. They are fun.