Lecture 3.2: Induction, and Strong Induction

CS 250, Discrete Structures, Fall 2015

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Adopted from previous lectures by Cinda Heeren, Zeph Grunschlag

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Course Admin

- Mid-Term 1
  - Graded; to be returned back today
  - Solution has been provided
- HW2
  - Solution will be provided very soon
  - We are grading it
Outline

- More practice: induction
- Strong Induction

Induction: some more exercises (Rosen)

- Show that $n < 2^n$, for $n=0, 1, 2, ...$

- Prove (generalized De Morgan’s)

$$
\bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} A_i
$$
Slightly harder exercise

Prove that if a set $S$ has $|S| = n$, then $|P(S)| = 2^n$

Base case ($n=0$): $S=\emptyset$, $P(S) = \{\emptyset\}$ and $|P(S)| = 1 = 2^0$

Assume $P(k)$: If $|S| = k$, then $|P(S)| = 2^k$

Prove that if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$

$S' = S U \{a\}$ for some $S \subseteq S'$ with $|S| = k$, and $a \in S'$.

Partition the power set of $S'$ into the sets containing $a$ and those not.

We count these sets separately.

Slightly harder exercise (contd)

Assume $P(k)$: If $|S| = k$, then $|P(S)| = 2^k$

Prove that if $|S'| = k+1$, then $|P(S')| = 2^{k+1}$

$S' = S U \{a\}$ for some $S \subseteq S'$ with $|S| = k$, and $a \in S'$.

Partition the power set of $S'$ into the sets containing $a$ and those not.

$P(S') = \{X : a \in X\} U \{X : a \notin X\}$

$P(S') = \{X : a \in X\} U P(S)$

Since these are all the subsets of elements in $S$.

Subsets containing $a$ are made by taking any set from $P(S)$, and inserting an $a$. 

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Slightly harder exercise (contd)

Assume P(k): If \(|S| = k\), then \(|P(S)| = 2^k\)

Prove that if \(|S'| = k+1\), then \(|P(S')| = 2^{k+1}\)

\(S' = S \cup \{a\}\) for some \(S \subseteq S'\) with \(|S| = k\), and \(a \in S'\).

\[
P(S') = \{X : a \in X\} \cup \{X : a \notin X\}
\]

Subsets containing \(a\) are made by taking any set from \(P(S)\), and inserting an \(a\).

\[
|P(S')| = |\{X : a \in X\}| + |P(S)|
\]

So \( |\{X : a \in X\}| = |P(S)| \)

\[
= 2 \cdot |P(S)| = 2 \cdot 2^k = 2^{k+1}
\]

Example

Recall the Fibonacci sequence:

\(\{f_n\} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\)

defined by \(f_0 = 0\), \(f_1 = 1\), and for \(n > 1\)

\(f_n = f_{n-1} + f_{n-2}\).

Notice that every third Fibonacci number is even:

LEMMA: For all natural numbers \(n\), \(2 | f_{3n}\).
Example

Proof. Base case $n = 0$.

$f_{3 \cdot 0} = f_0 = 0$ which is divisible by 2

Induction step, $n > 0$:

Assume $F_{3k}$ is divisible by 2

$F_{3k+3} = f_{3k+2} + f_{3k+1} = (f_{3k+1} + f_{3k}) + f_{3k+1}$

$= 2f_{3k+1} + f_{3k}$

By hypothesis, $2 | f_{3k}$ therefore

$2 | (2f_{3k+1} + f_{3k})$, so $2 | f_{3k+3}$, and the proof is complete.

Strong Mathematical Induction

If

- $P(0)$ and
- $\forall k \geq 0 \ (P(0) \land P(1) \land \ldots \land P(k)) \rightarrow P(k+1)$

Then

- $\forall n \geq 0 \ P(n)$

In our proofs, to show $P(k+1)$, our inductive hypothesis assures that ALL of $P(0), P(1), \ldots, P(k)$ are true, so we can use ANY of them to make the inference.
Strong Induction Example

Sometimes a stronger version of induction is needed, one that allows us to go back to smaller values than just the previous value of $n$. E.g. consider the Fibonacci sequence vs. the sequence $2^n$:

$$\{f_n\} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34$$
$$\{2^n\} = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512$$

**LEMMA:** For all $n$, $f_n < 2^n$

**Proof.**

- **Base $n = 0$:** $f_0 = 0 < 1 = 2^0$

- **Induction $n > 0$:**
  - Assume $f_k < 2^k$
  - Now, $f_{k+1} = f_k + f_{k-1} < 2^k + f_{k-1}$

Q: Now what?
Strong Induction Example

A: Would want to apply same formula to $k-1$. But strictly speaking, can’t because induction hypothesis only let us look at previous domino. This limitation on induction need not be so: If we could assume that the first $k$ dominoes falling implies that the $k+1$ domino falls, would be able to reduce back to smaller values, as needed here.

Strong induction formalizes this ability.

Strong Induction Example

So now, we can complete stuck proof:

**LEMMA:** For all $n$, $f_n < 2^n$

**Proof.** Base cases (both needed as can’t apply induction step on $f_1$ since $f_{-1}$ is undefined)

$n = 0$: $f_0 = 0 \times 1 = 2^0 \checkmark$

$n = 1$: $f_1 = 1 \times 2 = 2^1 \checkmark$
Strong Induction
Completing Example

So now, we can complete stuck proof:

**LEMMA:** For all \( n \), \( f_n < 2^n \)

**Proof.** Base cases (both needed as can’t apply
induction step on \( f_1 \) since \( f_{-1} \) is undefined)

- \( n = 0 \): \( f_0 = 0 < 1 = 2^0 \checkmark \)
- \( n = 1 \): \( f_1 = 1 < 2 = 2^1 \checkmark \)

Induction \( n > 0 \):

\[ f_{k+1} = f_k + f_{k-1} < 2^k + 2^{k-1} \]

applying both \( P(k) \) and \( P(k-1) \)
which can be assumed by strong induction
hypothesis. Doing more algebra:

\[ 2^k + 2^{k-1} = 2^k(1 + \frac{1}{2}) = 1.5*2^k < 2*2^k = 2^{k+1} \]

Therefore, \( f_{k+1} < 2^{k+1} \)

Another Example (Rosen)

- Prove that every integer > 1 can be
expressed as a product of prime numbers
Today’s Reading

- Rosen 5.2