Course Admin

- Mid-Term 2 Exam
  - Solution will be posted soon
  - Should have the results this week
- HW3
  - Solution will be posted soon
  - Should have the results next week
- HW4 will be posted by this week
  - Due in about 2 weeks
- Will have a 20-pointer bonus problem
Final Exam

- Tuesday, **December 8, 10:45am-1:15pm**, lecture room
  - Please mark the date/time/place
  - Cumulative coverage
- Our last lecture will be on December 3
  - We plan to do a final exam review then
- Note next week is off due to Fall Break/Thanksgiving.

Outline

- Closures
- Equivalence Relations
Closure

Consider relation \( R = \{(1,2),(2,2),(3,3)\} \) on the set \( A = \{1,2,3,4\} \).

Is \( R \) reflexive? **No**

What can we add to \( R \) to make it reflexive? \((1,1), (4,4)\)

\( R' = R \cup \{(1,1),(4,4)\} \) is called the reflexive closure of \( R \).

Closure

**Definition:**

The closure of relation \( R \) on set \( A \) with respect to property \( P \) is the relation \( R' \) with

1. \( R \subseteq R' \)
2. \( R' \) has property \( P \)
3. \( \forall S \text{ with } R \subseteq S \text{ and } S \text{ has property } P, R' \subseteq S \).
Reflexive Closure

Let \( r(R) \) denote the reflexive closure of relation \( R \).

Then \( r(R) = R \cup \{(a,a) : \forall a \in A\} \)

Fine, but does that satisfy the definition?

1. \( R \subseteq r(R) \) \hspace{1cm} \text{We added edges!}
2. \( r(R) \) is reflexive \hspace{1cm} \text{By defn}
3. Need to show that for any \( S \) with particular properties, \( r(R) \subseteq S \).

Let \( S \) be such that \( R \subseteq S \) and \( S \) is reflexive. Then \( \{(a,a) : \forall a \in A\} \subseteq S \) (since \( S \) is reflexive) and \( R \subseteq S \) (given). So, \( r(R) \subseteq S \).

Symmetric Closure

Let \( s(R) \) denote the symmetric closure of relation \( R \).

Then \( s(R) = R \cup \{(b,a) : (a,b) \in R\} \)

Fine, but does that satisfy the definition?

1. \( R \subseteq s(R) \) \hspace{1cm} \text{We added edges!}
2. \( s(R) \) is symmetric \hspace{1cm} \text{By defn}
3. Need to show that for any \( S \) with particular properties, \( s(R) \subseteq S \).

Let \( S \) be such that \( R \subseteq S \) and \( S \) is symmetric. Then \( \{(b,a) : (a,b) \in R\} \subseteq S \) (since \( S \) is symmetric) and \( R \subseteq S \) (given). So, \( s(R) \subseteq S \).
Transitive Closure

Let $t(R)$ denote the transitive closure of relation $R$.
Then $t(R) = R \cup \{(a,c) : \exists b (a,b),(b,c) \in R\}$

Example: $A=\{1,2,3,4\}, R=\{(1,2),(2,3),(3,4)\}$.
Apply definition to get:

$t(R) = \{(1,2),(2,3),(3,4), (1,3), (2,4)\}$

Which of the following is true:

a) This set is transitive, but we added too much.
b) This set is the transitive closure of $R$.
c) This set is not transitive, one pair is missing.
d) This set is not transitive, more than 1 pair is missing.

Transitive Closure

So how DO we find the transitive closure?

Example: $A=\{1,2,3,4\}, R=\{(1,2),(2,3),(3,4)\}$.

Define a path in a relation $R$, on $A$ to be a sequence of elements from $A$: $a,x_1,...,x_{i-1},x_i,...,x_{n-1},b$,
with $(a, x_i) \in R$, $\forall i (x_i,x_{i+1}) \in R$, $(x_{n-1},b) \in R$.
Transitive Closure

Formally:
If $t(R)$ is the transitive closure of $R$, and if $R$ contains a path from $a$ to $b$, then $(a,b) \in t(R)$

A technique:
- For a set $R$ consisting of $n$ elements, $t(R)$ can be specified by the matrix: $M_R \ V \ M_R^2 \ V \ ... \ V \ M_R^n$
- More efficient method: Warshall's algorithm

Transitive Closure -- Example

- $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
- $M_R \ V \ M_R^2 \ V \ ... \ V \ M_R^n$?
Equivalence Relations

Example:
Let $S = \{\text{people in this classroom}\}$, and let $R = \{(a,b): a \text{ has same # of coins in his/her bag as } b\}$

Quiz time:
Is R reflexive? Yes
Is R symmetric? Yes
Is R transitive? Yes

This is a special kind of relation, characterized by the properties it has.

Everyone with the same # of coins as you is just like you.

Equivalence Relations

Formally:
Relation R on A is an equivalence relation if R is
- Reflexive ($\forall a \in A, aRa$)
- Symmetric ($aRb \rightarrow bRa$)
- Transitive ($aRb \text{ AND } bRc \rightarrow aRc$)

Example:
$S = \mathbb{Z}$ (integers), $R = \{(a,b): a \equiv b \mod 4\}$

Is this relation an equivalence relation on S?
Have to PROVE reflexive, symmetric, transitive.

Recall: $aRb$ denotes $(a,b) \in R$. 
Equivalence Relations

Example:

\[ S = \mathbb{Z} \text{ (integers)}, \ R = \{(a,b) : a \equiv b \mod 4\} \]

Is this relation an equivalence relation on \( S \)?

Start by thinking of \( R \) a different way: \( aRb \) iff there is an int \( k \) so that \( a = 4k + b \). Your quest becomes one of finding \( k \).

Let \( a \) be any integer. \( aRa \) since \( a = 4 \cdot 0 + a \).

Consider \( aRb \). Then \( a = 4k + b \). But \( b = -4k + a \).

Consider \( aRb \) and \( bRc \). Then \( a = 4k + b \) and \( b = 4j + c \).

So, \( a = 4k + 4j + c = 4(k+j) + c \).

Equivalence Relations

Example:

- \( S = \mathbb{Z} \text{ (integers)}, \ R = \{(a,b) : a = b \text{ or } a = -b\} \).
  - Is this relation an equivalence relation on \( S \)?
  - Have to prove reflexive, symmetric, transitive.
Equivalence Relations

Example:

- $S = \mathbb{R}$ (real numbers), $R = \{(a,b) : a - b$ is an integer$\}$. Is this relation an equivalence relation on $S$?
- Have to prove reflexive, symmetric, transitive.
Equivalence Relations

- Example:
  - $S = \mathbb{N}$ (natural numbers), $R = \{(a,b) : a \mid b \}$. Is this relation an equivalence relation on $S$?
  - Have to prove reflexive, symmetric, transitive.

Equivalence Classes

Example:
Back to coins in bags.

Definition: Let $R$ be an equivalence relation on $S$.
The equivalence class of $a \in S$, $[a]_R$, is
$[a]_R = \{b : aRb\}$

$a$ is just a name for the equiv class. Any member of the class is a representative.
**Equivalence Classes**

**Definition:** Let $R$ be an equivalence relation on $S$. The *equivalence class* of $a \in S$, $[a]_R$, is

$$[a]_R = \{b : aRb\}$$

What does the set of equivalence classes on $S$ look like?

To answer, think about the relation from before:

- $S = \{\text{people in this room}\}$
- $R = \{(a,b) : a \text{ has the same # of coins in his/her bag as } b\}$

In how many different equivalence classes can each person fall?

Similarly, consider the $a \equiv b \mod 4$ relation

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**Today’s Reading**

- Rosen 9.4 and 9.5